Three 3D Problems that Absolutely Need Diagrams. Lots of them.

1. For a cuboid with dimensions 5cm, 7cm and 10cm,
a) Work out the lengths of its face diagonals
b) Work out the lengths of its space diagonals
c) Work out the angle between the face diagonal of the largest face and the space diagonal that connects to it.
d) Work out the cross-sectional areas, if the cuboid is sliced from one edge to the diagonally opposite edge.

2. For a rectangular based pyramid with base dimensions 8cm and 11cm, and slanting edge length of 15cm
a) Work out the height of the pyramid
b) Work out the angle that a slanting edge makes with the rectangular base
c) Work out the areas of the slanting faces
d) Work out the angles that the slanting faces make with the rectangular base.
e) Work out the largest cross-sectional area possible if the pyramid is sliced vertically through its apex
f) If the pyramid is sliced horizontally half way up its height, what percentage of its overall volume is in the top portion?

3. In a cube of side length 7cm, work out the angle between the space diagonal and a face diagonal.

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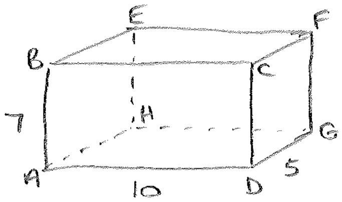
3D Problems - Solutions.

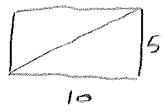
1.

a)



in all applications of pythagoras' theorem, we shall take the positive solution, as all lengths >0.

Face diagonal of ADGH/BCFE



length $= \sqrt{10^2+5^2}$

$= \sqrt{125}$

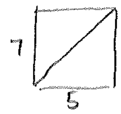
$= \sqrt{25\times5}$

$= 5\sqrt{5}$

$= 11{\cdot}1803...$

$= 11{\cdot}2$cm (1dp)

Face diagonal of ABEH/DEFG



length $= \sqrt{7^2+5^2}$

$= \sqrt{49+25}$

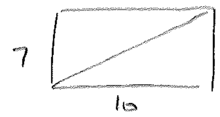
$= \sqrt{74}$

$= \sqrt{2}\sqrt{37}$

$= 8{\cdot}60233..$

$= 8{\cdot}6$cm (1dp)

Face Diagonal ABCD/EFGH



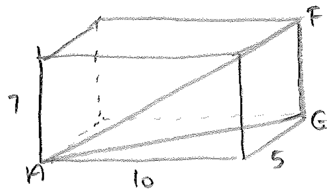
length $= \sqrt{7^2+10^2}$

$= \sqrt{149}$

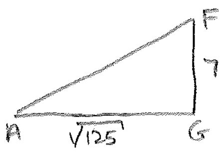
$= 12{\cdot}2066...$

$= 12{\cdot}2$ cm (1dp)

b)



$\Rightarrow$



$\Rightarrow$ $AF^2 = 7^2 + (\sqrt{125})^2$

$= 49 + 125$

$= 174$

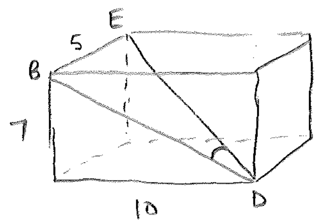
$AF = \sqrt{174}$

$= 13{\cdot}1909...$

$= 13{\cdot}2$cm (1dp)

All space diagonals of a cuboid are the same length. ☺

c)



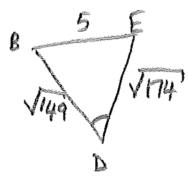
BD is face diagonal & has length $\sqrt{149}$

ED is space diagonal & has length $\sqrt{174}$

BE is edge & has length 5

angle wanted is BDE

so we have



so $\cos D = \dfrac{b^2+e^2-d^2}{2be}$ by cosine rule

$= \dfrac{174+149-5^2}{2\times\sqrt{174}\times\sqrt{149}}$

$= \dfrac{298}{2\sqrt{174}\sqrt{149}}$

$= \dfrac{149}{\sqrt{174}\sqrt{149}}$ aha! Rationalise denominator as 149's are everywhere!

now $\cos D = \dfrac{149}{\sqrt{174}\sqrt{149}} \times \dfrac{\sqrt{149}}{\sqrt{149}}$ (to rationalise denominator)

$$= \frac{\cancel{149}\sqrt{149}}{\sqrt{174} \times \cancel{149}}$$

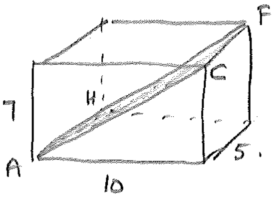
$$= \frac{\sqrt{149}}{\sqrt{174}}$$

so $D = \cos^{-1}\left(\sqrt{\dfrac{149}{174}}\right)$

$= 22.2748\ldots$

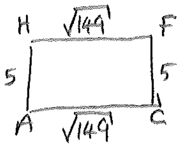
$= \underline{\underline{22.3^\circ}}$ (1dp)

d)



Sliced plane is ACFH

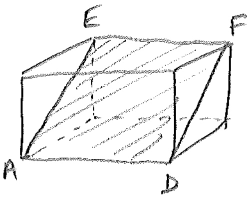
This is a rectangle



$\Rightarrow$ Area $= 5\sqrt{149}$

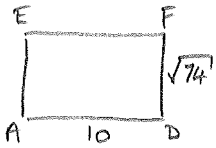
$= 61.0328\ldots$

$= \underline{\underline{61.0\text{ cm}^2}}$. (1dp)



slice plane is ADEF

This is a rectangle

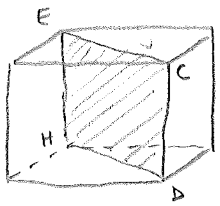


$\Rightarrow$ Area $= 10\sqrt{74}$

$= 10 \times 8.60233\ldots$

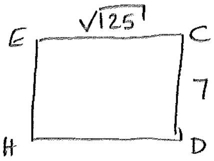
$= 86.0233\ldots$

$= \underline{\underline{86.0\text{ cm}^2}}$ (1dp)



slice plane is CDHE

This is a rectangle



$\Rightarrow$ Area $= 7\sqrt{125}$

$= 7 \times 5\sqrt{5}$

$= 35\sqrt{5}$

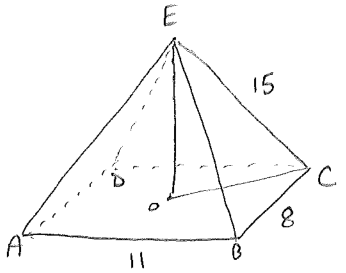
$= 78.2624\ldots$

$= \underline{\underline{78.3\text{ cm}^2}}$ (1dp)

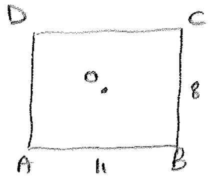
3D Problems - Solutions

2.

a)

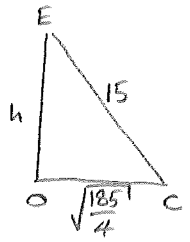


point O is in the centre of base ABCD



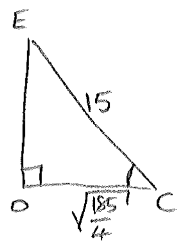
$$OC = \frac{1}{2}AC$$
$$= \frac{1}{2}\sqrt{11^2+8^2}$$
$$= \frac{1}{2}\sqrt{121+64}$$
$$= \frac{1}{2}\sqrt{185}$$
$$\text{so } OC^2 = \left(\frac{1}{2}\sqrt{185}\right)^2$$
$$= \frac{1}{4} \times 185.$$
$$= \frac{185}{4}$$

consider triangle OEC:



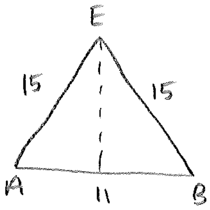
$$\text{so } h^2 + \left(\frac{185}{4}\right) = 15^2$$
$$h^2 = 15^2 - \frac{185}{4}$$
$$h^2 = \frac{715}{4}$$
$$h = \sqrt{\frac{715}{4}}$$
$$h = 13 \cdot 3697\ldots$$
$$\text{height} = \underline{\underline{13 \cdot 4 \text{ cm}}} \ (1\text{dp})$$

b)

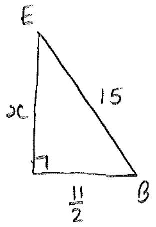


$$\text{So } \cos C = \frac{\text{adj}}{\text{hyp}}$$
$$\cos C = \frac{\sqrt{\frac{185}{4}}}{15}$$
$$\cos C = 0 \cdot 453382\ldots$$
$$C = \cos^{-1}(0 \cdot 453382\ldots)$$
$$C = 63 \cdot 0391\ldots$$
$$\underline{\underline{C = 63 \cdot 0^\circ}} \ (1\text{dp})$$

c)



$\rightarrow$



$\rightarrow$

$$x^2 + \left(\tfrac{11}{2}\right)^2 = 15^2$$

$$x^2 + \frac{121}{4} = 225$$

$$x^2 = \frac{779}{4}$$

$$x = \sqrt{\frac{779}{4}}$$

$$x = 13{\cdot}9553\ldots$$

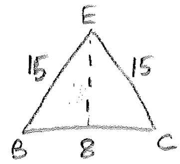
so Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 11 \times x$

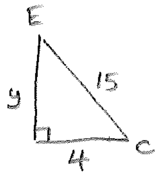
$= 76{\cdot}7541\ldots$

$= 76{\cdot}8 \text{ cm}^2.$

other slanting face :



$\rightarrow$



$\rightarrow$

$$y^2 + 4^2 = 15^2$$

$$y^2 = 225 - 16$$

$$y^2 = 209$$

$$y = \sqrt{209}$$

So area $= \frac{1}{2} \times \text{base} \times \text{height}$

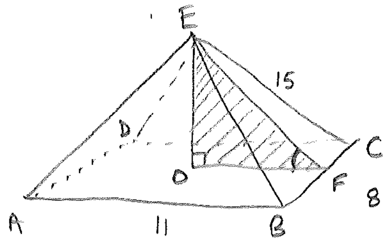
$= \frac{1}{2} \times 8 \times \sqrt{209}$

$= 4\sqrt{209}$

$= 57{\cdot}8273\ldots$

$\approx 57{\cdot}8 \text{ cm}^2$ (1dp)

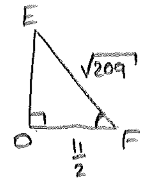
d)



now length EF worked out previously to be $\sqrt{209}$

length OF is $\frac{11}{2}$

so



so $\cos F = \frac{adj}{hyp}$

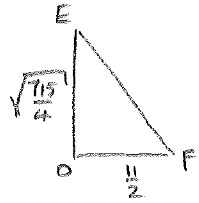
$\cos F = \frac{11/2}{\sqrt{209}}$

$F = \cos^{-1}(0.380443..)$

$F = 67.6389...$

$F = 67.6^\circ$ (1dp)

OR



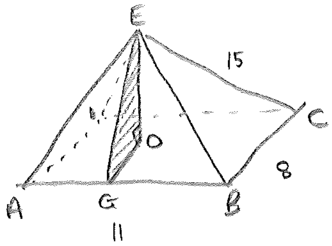
so $\tan F = \frac{\sqrt{\frac{715}{4}}}{11/2}$

$F = \tan^{-1}(2.43)$

$F = 67.6389...$

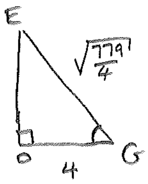
etc

etc.



now length EG worked out previously to be $\sqrt{\frac{779}{4}}$

so



so $\cos G = \frac{adj}{hyp}$.

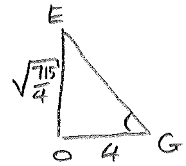
$\cos G = \frac{4}{\sqrt{\frac{779}{4}}}$

$G = \cos^{-1}(0.28663..)$

$G = 73.3437...$

$G = 73.3^\circ$ (1dp)

OR



$\tan G = \frac{\sqrt{\frac{715}{4}}}{4}$

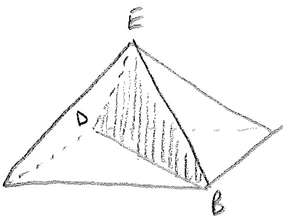
$G = \tan^{-1}(3.34244)$

$G = 73.3437$

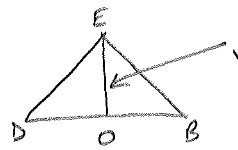
etc

etc.

e)



Slice through BDE



$\sqrt{\frac{715}{4}}$ from part A

length $DB = \sqrt{185}$ from part (a)

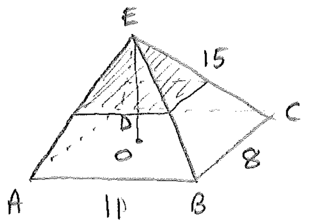
so Area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times \sqrt{185} \times \sqrt{\frac{715}{4}}$

$= 90.9241...$

$= 90.9 \text{ cm}^2$ (1dp)

f)



$OE = \sqrt{\frac{715}{4}}$

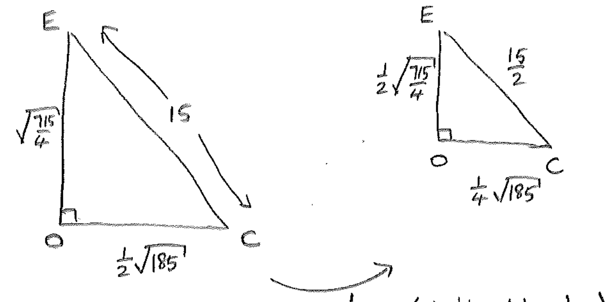
Volume whole pyramid $= \frac{1}{3} \times \text{base area} \times \text{height}$

$= \frac{1}{3} \times 11 \times 8 \times \sqrt{\frac{715}{4}}$

$= 392.179\ldots$

$= \underline{\underline{392.2\,\text{cm}^3}}$ (1dp)

If chopped off half way down :



$\times \frac{1}{2}$ (similar triangles)

so new height is $\frac{1}{2}\sqrt{\frac{715}{4}}$

new base has diagonal length (OC) to be half of what it was

so all lengths of base have been halved

$\Rightarrow$ base area now $\left(\frac{1}{2}\right)^2$ of what it was

so Volume top section $= \frac{1}{3} \times \text{base area} \times \text{height}$

$= \frac{1}{3} \times \underbrace{\left(\tfrac{1}{2}\right)^2 \times 11 \times 8}_{\text{new base area}} \times \underbrace{\tfrac{1}{2}\sqrt{\tfrac{715}{4}}}_{\text{new height}}$

$= \left(\frac{1}{2}\right)^3 \times \text{original volume}$

$= 49.0224$

$= 49.0\,\text{cm}^3$ (1dp)

So % in top portion $= \dfrac{49.0224}{392.179}$

$= 0.125$

$= \underline{\underline{12.5\,\%}}$

OR

$\dfrac{\text{top}}{\text{whole}} = \dfrac{\left(\frac{1}{2}\right)^3 \times \text{whole}}{\text{whole}}$

$= \left(\frac{1}{2}\right)^3$

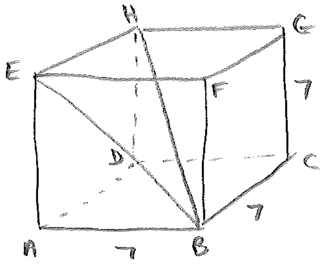
$= \frac{1}{8}$

$= 0.125$

$= \underline{\underline{12.5\,\%}}$

3D Problems – Solutions.

3.



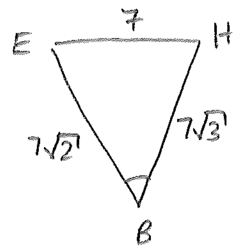
Face diagonal is EB

Space diagonal is HB.

EB has length $\sqrt{7^2+7^2}$
$= \sqrt{2\times 7^2}$
$= \sqrt{2}\sqrt{7^2}$
$= \sqrt{2}\times 7$
$= 7\sqrt{2}$

HB has length $\sqrt{7^2+7^2+7^2}$
$= 7\sqrt{3}$

So



$$\cos B = \frac{e^2+h^2-b^2}{2eh}$$ by cosine rule

$$\cos B = \frac{(7\sqrt{2})^2+(7\sqrt{3})^2-7^2}{2.7\sqrt{2}.7\sqrt{3}}$$

$$\cos B = \frac{49\times2+49\times3-49}{2\times49\times\sqrt{6}}$$

$$\cos B = \frac{\cancel{49}\times4}{2\times\cancel{49}\times\sqrt{6}}$$

$$\cos B = \frac{2}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$ to rationalise denominator

$$\cos B = \frac{2\sqrt{6}}{6}$$

$$\cos B = \frac{\sqrt{6}}{3}.$$

$$B = \cos^{-1}\left(\frac{\sqrt{6}}{3}\right)$$

$$B = 35.2644\ldots$$

$$\underline{\underline{B = 35.3^\circ}} \text{ (1dp)}$$